

THE SMALLEST LINE ARRANGEMENT WHICH IS FREE BUT NOT RECURSIVELY FREE

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ABSTRACT. In the category of free arrangements, inductively and recursively free arrangements are important. In particular, in the former, the conjecture by Terao asserting that freeness depends only on combinatorics holds true. A long standing problem whether all free arrangements are recursively free or not is settled by Cuntz and Hoge very recently, by giving a free but non-recursively free plane arrangement consisting of 27 planes.

In this paper, we construct a free but non-recursively free plane arrangement consisting of 13 planes, and show that this example is the smallest in the sense of the cardinality of planes. In other words, all free plane arrangements consisting of at most 12 planes are recursively free. To show it, we completely classify all free plane arrangements in terms of inductive freeness and three exceptions when the number of planes is at most 12.

1. INTRODUCTION

In the study of hyperplane arrangements, one of the most important problems is the freeness of them. In general, to determine whether a given arrangement is free or not is very difficult, and there is essentially only one way to check it, Saito's criterion (Theorem 2.1). On the other hand, there is a nice way to construct a free arrangement from a given free arrangement, called the addition-deletion theorem (Theorem 2.4). Since the empty arrangement is free, there is a natural question whether every free arrangement can be obtained, starting from empty arrangement, by applying addition and deletion theorems. For simplicity, for the rest of this section, let us concentrate our interest on the central arrangements in \mathbb{C}^3 .

We say that a central arrangement \mathcal{A} is *inductively free* if it can be constructed by using only the addition theorem from the empty arrangement, and *recursively free* if we use both the addition and deletion theorems to construct it. It was very soon to be found a free arrangement which is not inductively free (see Example 4.59 in [OT] for example). However, a free but non-recursively free arrangement has not been found for a long time. It was very recent that Cuntz and Hoge first found such an example in [CH], which consists of 27 planes over $\mathbb{Q}(\zeta)$ with the fifth root of unity ζ in \mathbb{C} .

The aim of this paper is to give a new example of free but non-recursively free arrangements in \mathbb{C}^3 consisting of the smallest number of planes. Our example consists of 13 planes defined over $\mathbb{Q}[\sqrt{3}]$.¹ To show that there are no such arrangements when the number of planes is strictly less than 13, we also investigate the set of all free arrangements \mathcal{A} with $|\mathcal{A}| \leq 12$. In other words, we give the complete classification of such free arrangements in terms of inductive, recursive freeness and three exceptions given in Definitions 3.2, 4.2 and 5.2. Now let us state our main theorem in the following.

Theorem 1.1. *Let \mathcal{A} be a central arrangement in \mathbb{C}^3 .*

The first author's work was partially supported by Japan Society for the Promotion of Science Grant-in-Aid for Young Scientists (B), No. 24740012.

The second author's work was partially supported by Japan Society for the Promotion of Science Grant-in-Aid for Young Scientists (B), No. 23740016.

¹ After posting the first version of this paper, we were informed by Professor Cuntz that he also found this example independently. See [C].

(1) If \mathcal{A} is free with $|\mathcal{A}| \leq 12$, then \mathcal{A} is recursively free. More precisely, \mathcal{A} is either inductively free or one of the following arrangements characterized by their lattice structures.

- (i) A dual Hesse arrangement, i.e., the arrangement \mathcal{A} with $|\mathcal{A}| = 9$, whose set $L_2(\mathcal{A})$ of codimension 2 intersections consists of 12 triple lines.
- (ii) A pentagonal arrangement, i.e., the arrangement \mathcal{A} with $|\mathcal{A}| = 11$ such that $L_2(\mathcal{A})$ consists of 10 double lines, 5 triple lines, 5 quadruple lines and that any $H \in \mathcal{A}$ contains at most 5 lines of $L_2(\mathcal{A})$.
- (iii) A monomial arrangement associated to the group $G(4, 4, 3)$, i.e., the arrangement \mathcal{A} with $|\mathcal{A}| = 12$, whose $L_2(\mathcal{A})$ consists of 16 triple lines and 3 quadruple lines.

Moreover, the lattices of arrangements in (i), (ii) and (iii) are realized over $\mathbb{Q}[\sqrt{-3}]$, $\mathbb{Q}[\sqrt{5}]$ and $\mathbb{Q}[\sqrt{-1}]$, respectively. In particular, all free arrangements in \mathbb{C}^3 are inductively free when $|\mathcal{A}| \leq 8$ or $|\mathcal{A}| = 10$.

(2) There exists a free but not recursively free arrangement \mathcal{A} over $\mathbb{Q}[\sqrt{3}]$ with $|\mathcal{A}| = 13$.

Our proof is based on the combinatorial method. We can check Theorem 1.1 by easy computations by hand, or just drawing a nice picture of our arrangement.

Theorem 1.1 contains a characterization of free arrangements \mathcal{A} in \mathbb{C}^3 with $|\mathcal{A}| \leq 12$. There have been several researches on this way, especially from the viewpoint of the conjecture by Terao (Conjecture 2.3), which asks whether the freeness of an arrangement depends only on its combinatorics. For example, the conjecture by Terao was checked when $|\mathcal{A}| \leq 11$ in [WY], and $|\mathcal{A}| \leq 12$ in [FV]. By Theorem 1.1, we can give another proof of the conjecture by Terao when $|\mathcal{A}| \leq 12$ which is originally due to Faenzi and Vallès.

Corollary 1.2 (Faenzi-Vallès [FV]). *The conjecture by Terao holds for central arrangements \mathcal{A} in \mathbb{C}^3 with $|\mathcal{A}| \leq 12$.*

By investigating the structure of the classification in Theorem 1.1, we can say that almost all the free arrangements with small exponents are inductively free.

Corollary 1.3. *Let \mathcal{A} be a free arrangement in \mathbb{C}^3 with $\exp(\mathcal{A}) = (1, a, b)$. If $\min(a, b) \leq 4$, then \mathcal{A} is either inductively free or a dual Hesse arrangement appearing in Theorem 1.1.*

The organization of our paper is as follows. In §2 we introduce several definitions and results which will be used in this paper. In §3, §4 and §5 we prove Theorem 1.1 (1). In §6 we prove Theorem 1.1 (2).

Acknowledgements. We are grateful to M. Yoshinaga for his helpful comments.

2. PRELIMINARIES

In this section, we summarize several definitions and results which will be used in this paper. For the basic reference on the arrangement theory, we refer Orlik-Terao [OT].

Let $V = \mathbb{C}^n$. An arrangement of hyperplanes \mathcal{A} is a finite set of affine hyperplanes in V . An arrangement \mathcal{A} is *central* if every hyperplane is linear. For a hyperplane $H \subset V$, define

$$\mathcal{A} \cap H = \{H \cap H' \neq \emptyset \mid H' \in \mathcal{A}, H' \neq H\}.$$

Hence $\mathcal{A} \cap H$ is an arrangement in an $(n-1)$ -dimensional vector space H . Let us define a *cone* $c\mathcal{A}$ of an affine arrangement \mathcal{A} as follows. If \mathcal{A} is defined by a polynomial equation $Q = 0$, then $c\mathcal{A}$ is defined by $z \cdot cQ = 0$, where cQ is the homogenized polynomial of Q by the new coordinate z . When \mathcal{A} is central, let us fix a defining linear form $\alpha_H \in V^*$ for each $H \in \mathcal{A}$.

From now on, let us concentrate our interest on central arrangements in \mathbb{C}^n when $n = 2$ or 3. So arrangements of lines or planes. Even when $n = 3$, identifying \mathbb{C}^3 with $P_{\mathbb{C}}^2$, we also say they are line arrangements when there are no confusions. Let $S = S(V^*) =$

$\mathbb{C}[x_1, \dots, x_n]$ be the coordinate ring of V . For a central arrangement \mathcal{A} , define

$$D(\mathcal{A}) = \left\{ \theta \in \bigoplus_{i=1}^n S \partial / \partial x_i \mid \theta(\alpha_H) \in S \alpha_H \text{ for all } H \in \mathcal{A} \right\}.$$

$D(\mathcal{A})$ is called the *logarithmic derivation module*. $D(\mathcal{A})$ is reflexive, but not free in general. We say that \mathcal{A} is free with *exponents* (d_1, \dots, d_n) if $D(\mathcal{A})$ has a homogeneous free basis $\theta_1, \dots, \theta_n$ with $\deg \theta_i = d_i$ ($i = 1, \dots, n$). Here the *degree* of a homogeneous derivation $\theta = \sum_{i=1}^n f_i \partial / \partial x_i$ is defined by $\deg f_i$ for all non-zero f_i . Note that the Euler derivation $\theta_E = \sum_{i=1}^n x_i \partial / \partial x_i$ is contained in $D(\mathcal{A})$. In particular, it is easy to show that $D(\mathcal{A})$ has $S \theta_E$ as its direct summand for a non-empty arrangement \mathcal{A} . Hence $\exp(\mathcal{A})$ always contains 1 if \mathcal{A} is not empty.

To verify the freeness of \mathcal{A} , the following Saito's criterion is essential.

Theorem 2.1 (Saito's criterion, [S]). *Let $\theta_1, \dots, \theta_n \in D(\mathcal{A})$. Then the following three conditions are equivalent.*

- (1) \mathcal{A} is free with basis $\theta_1, \dots, \theta_n$.
- (2) $\det[\theta_i(x_j)] = c \prod_{H \in \mathcal{A}} \alpha_H$ for some non-zero $c \in \mathbb{C}$.
- (3) $\theta_1, \dots, \theta_n$ are all homogeneous derivations, S -independent and $\sum_{i=1}^n \deg \theta_i = |\mathcal{A}|$.

For a multiplicity $m: \mathcal{A} \rightarrow \mathbb{Z}_{>0}$, we can define the logarithmic derivation module $D(\mathcal{A}, m)$ of a multiarrangement (\mathcal{A}, m) by

$$D(\mathcal{A}, m) = \left\{ \theta \in \bigoplus_{i=1}^n S \partial / \partial x_i \mid \theta(\alpha_H) \in S \alpha_H^{m(H)} \text{ for all } H \in \mathcal{A} \right\}.$$

The freeness, exponents, and Saito's criterion for a multiarrangement can be defined in the same manner as an arrangement case. Note that the Euler derivation is not contained in $D(\mathcal{A}, m)$ in general.

Recall that every central (multi)arrangement in \mathbb{C}^2 is free since $\dim_{\mathbb{C}} \mathbb{C}^2 = 2$ and $D(\mathcal{A}, m)$ is reflexive. Hence the first non-free central arrangement occurs when $n = 3$. Define the *intersection lattice* $L(\mathcal{A})$ of \mathcal{A} by

$$L(\mathcal{A}) = \left\{ \bigcap_{H \in \mathcal{B}} H \mid \mathcal{B} \subset \mathcal{A} \right\}.$$

$L(\mathcal{A})$ has a poset structure with an order by the reverse inclusion. $L(\mathcal{A})$ is considered to be a combinatorial information of \mathcal{A} .

Define the *Möbius function* $\mu: L(\mathcal{A}) \rightarrow \mathbb{Z}$ by

$$\mu(V) = 1, \quad \mu(X) = - \sum_{Y \in L(\mathcal{A}), X \subsetneq Y \subset V} \mu(Y) \quad (X \neq V).$$

Then the *characteristic polynomial* $\chi(\mathcal{A}, t)$ of \mathcal{A} is defined by $\chi(\mathcal{A}, t) = \sum_{X \in L(\mathcal{A})} \mu(X) t^{\dim X}$. It is known that

$$\chi(\mathcal{A}, t) = t^n \text{Poin}(\mathbb{C}^n \setminus \cup_{H \in \mathcal{A}} H, -t^{-1}).$$

Hence $\chi(\mathcal{A}, t)$ is both combinatorial and topological invariants of an arrangement. The freeness and $\chi(\mathcal{A}, t)$ are related by the following formula.

Theorem 2.2 (Factorization, [T2]). *Assume that a central arrangement \mathcal{A} is free with $\exp(\mathcal{A}) = (d_1, \dots, d_n)$. Then*

$$\chi(\mathcal{A}, t) = \prod_{i=1}^n (t - d_i).$$

Theorem 2.2 implies that the algebra of an arrangement might control the combinatorics and the topology of it. However, the converse is not true in general. For example, there is a non-free arrangement in \mathbb{C}^3 whose characteristic polynomial factorizes over the ring

of integers (see [OT]). Hence it is natural to ask how much algebra and combinatorics of arrangements are related.

The following conjecture is one of the largest ones in the theory of arrangements.

Conjecture 2.3 (Terao). *The freeness of an arrangement depends only on the combinatorics.*

Conjecture 2.3 is open even when $n = 3$. In [T1], Terao introduced a nice family of free arrangements in which Conjecture 2.3 holds. To state it, let us introduce the following key theorem in this paper.

Theorem 2.4 (Addition-Deletion, [T1]). *Let \mathcal{A} be a free arrangement in \mathbb{C}^3 with $\exp(\mathcal{A}) = (1, d_1, d_2)$.*

- (1) *(the addition theorem). Let $H \notin \mathcal{A}$ be a linear plane. Then $\mathcal{B} = \mathcal{A} \cup \{H\}$ is free with $\exp(1, d_1, d_2 + 1)$ if and only if $|\mathcal{A} \cap H| = 1 + d_1$.*
- (2) *(the deletion theorem). Let $H \in \mathcal{A}$. Then $\mathcal{A}' = \mathcal{A} \setminus \{H\}$ is free with $\exp(1, d_1, d_2 - 1)$ if and only if $|\mathcal{A}' \cap H| = 1 + d_1$.*

Definition 2.5.

- (1) A central plane arrangement \mathcal{A} is *inductively free* if there is a filtration of subarrangements $\mathcal{A}_1 \subset \mathcal{A}_2 \subset \dots \subset \mathcal{A}_\ell = \mathcal{A}$ such that $|\mathcal{A}_i| = i$ ($1 \leq i \leq \ell$) and every \mathcal{A}_i is free.
- (2) A central plane arrangement \mathcal{A} is *recursively free* if there is a sequence of arrangements $\emptyset = \mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_t = \mathcal{A}$ such that $||\mathcal{B}_{i+1}| - |\mathcal{B}_i|| = 1$ ($1 \leq i \leq t - 1$) and every \mathcal{B}_i is free.

Roughly speaking, an inductively free arrangement is a free arrangement constructed from an empty arrangement by using only the addition theorem, and a recursively free arrangement is a free arrangement constructed from an empty arrangement by using both the addition and deletion theorems. It is known that in the category of inductively free arrangements, Conjecture 2.3 is true, but open in that of recursively free arrangements.

Remark 2.6. Theorem 2.4 and Definition 2.5 are different from those in an arbitrary dimensional case. They coincide when $n = 3$ because every arrangement in \mathbb{C}^2 is free. For a general definition, see [T1] and [OT] for example.

Definition 2.7. For $\ell \in \mathbb{Z}_{\geq 0}$, we define the sets \mathcal{F}_ℓ , \mathcal{I}_ℓ and \mathcal{R}_ℓ as follows.

$$\begin{aligned} \mathcal{F}_\ell &= \{\text{free arrangement } \mathcal{A} \text{ in } \mathbb{C}^3 \text{ with } |\mathcal{A}| = \ell\}, \\ \mathcal{I}_\ell &= \{\mathcal{A} \in \mathcal{F}_\ell \mid \mathcal{A} \text{ is inductively free}\}, \\ \mathcal{R}_\ell &= \{\mathcal{A} \in \mathcal{F}_\ell \mid \mathcal{A} \text{ is recursively free}\}. \end{aligned}$$

Note that $\mathcal{I}_\ell \subset \mathcal{R}_\ell \subset \mathcal{F}_\ell$ by definition.

For the rest of this section, we concentrate our interest on a central arrangements in \mathbb{C}^3 . One of the main purposes of this paper is to clarify the difference between \mathcal{I}_ℓ and \mathcal{F}_ℓ for $0 \leq \ell \leq 12$. Since $\mathcal{I}_\ell = \mathcal{F}_\ell$ for $\ell \leq 1$, we may assume $\ell \geq 2$. For $H \in \mathcal{A}$, we denote $n_{\mathcal{A},H} = |\mathcal{A} \cap H|$. The following is the foundation stone of our analysis in this paper.

Theorem 2.8 ([A]). *Assume $\chi(\mathcal{A}, t) = (t - 1)(t - a)(t - b)$ for $a, b \in \mathbb{R}$ and there exists $H \in \mathcal{A}$ such that $n_{\mathcal{A},H} > \min(a, b)$. Then, \mathcal{A} is free if and only if $n_{\mathcal{A},H} \in \{a + 1, b + 1\}$.*

Definition 2.9. In view of theorem 2.8, we define the subset \mathcal{S}_ℓ of \mathcal{F}_ℓ as

$$\mathcal{S}_\ell = \{\mathcal{A} \in \mathcal{F}_\ell \mid \exp(\mathcal{A}) = (1, a, b), \max_{H \in \mathcal{A}} n_{\mathcal{A},H} \leq \min(a, b)\}.$$

Note that $\mathcal{S}_\ell \cap \mathcal{I}_\ell = \emptyset$ by Theorem 2.4.

Lemma 2.10. *If $\mathcal{F}_{\ell-1} = \mathcal{I}_{\ell-1}$ (resp. $\mathcal{R}_{\ell-1}$), then $\mathcal{F}_\ell = \mathcal{S}_\ell \sqcup \mathcal{I}_\ell$ (resp. $\mathcal{S}_\ell \cup \mathcal{R}_\ell$).*

Proof. Take $\mathcal{A} \in \mathcal{F}_\ell \setminus \mathcal{S}_\ell$ and set $\exp(\mathcal{A}) = (1, a, b)$. By Theorem 2.8, $\mathcal{A} \in \mathcal{F}_\ell \setminus \mathcal{S}_\ell$ if and only if there exists a line $H \in \mathcal{A}$ such that $n_{\mathcal{A},H} = a + 1$ or $b + 1$. Therefore, by the deletion theorem, we have $\mathcal{A} \setminus \{H\} \in \mathcal{F}_{\ell-1}$. Now the assertions above are clear. \square

In the rest of this paper, we regard a central arrangement in \mathbb{C}^3 as a line arrangement in $\mathbb{P}_{\mathbb{C}}^2$. For a line arrangement \mathcal{A} in $\mathbb{P}_{\mathbb{C}}^2$ and $P \in \mathbb{P}_{\mathbb{C}}^2$, we set

$$\mathcal{A}_P = \{H \in \mathcal{A} \mid P \in H\}, \mu_P(\mathcal{A}) = |\mathcal{A}_P| - 1, \mu_{\mathcal{A}} = \sum_{P \in \mathbb{P}_{\mathbb{C}}^2} \mu_P(\mathcal{A}).$$

Note that $\mu_P(\mathcal{A})$ is the reformulation of Möbius function for $L_2(\mathcal{A})$. If $\mathcal{A} \neq \emptyset$, we can express $\chi(\mathcal{A}, t)$ as follows by definition.

$$\chi(\mathcal{A}, t) = (t-1) \left\{ t^2 - (|\mathcal{A}| - 1)(t+1) + \mu_{\mathcal{A}} \right\}.$$

Concerning the set \mathcal{S}_{ℓ} , we have the following lemma.

Lemma 2.11. *Let $\mathcal{A} \in \mathcal{F}_{\ell}$ with $\ell \geq 2$ and $\exp(\mathcal{A}) = (1, a, b)$. Assume $\mu_P(\mathcal{A}) \geq \min(a, b) - 1$ for some $P \in \mathbb{P}_{\mathbb{C}}^2$. Then we have $\mathcal{A} \notin \mathcal{S}_{\ell}$. In particular, $\mathcal{S}_{\ell} = \emptyset$ for $2 \leq \ell \leq 6$.*

Proof. We assume $a \leq b$ and $\mathcal{A} \in \mathcal{S}_{\ell}$. Let P_0 be a point in $\mathbb{P}_{\mathbb{C}}^2$.

Suppose $\mu_{P_0}(\mathcal{A}) \geq a$. If there exists $H \in \mathcal{A} \setminus \mathcal{A}_{P_0}$, we have $n_{\mathcal{A}, H} \geq |\mathcal{A}_{P_0} \cap H| \geq a + 1$, which contradicts to $\mathcal{A} \in \mathcal{S}_{\ell}$. Therefore $\mathcal{A}_{P_0} = \mathcal{A}$, and it follows that $\exp(\mathcal{A}) = (1, 0, \ell - 1)$. Thus $a = 0$, but there exists $H \in \mathcal{A}$ with $n_{\mathcal{A}, H} \geq 1$ since $\ell \geq 2$, which contradicts to $\mathcal{A} \in \mathcal{S}_{\ell}$.

Suppose $\mu_{P_0}(\mathcal{A}) = a - 1$. Since $|\mathcal{A}_{P_0}| = a$ and $\mathcal{A} \in \mathcal{S}_{\ell}$, all intersection points of \mathcal{A} lie on $\bigcup_{H \in \mathcal{A}_{P_0}} H$. It follows that \mathcal{A} is super solvable (see Definition 2.32 of [OT] for the details) and $\exp(\mathcal{A}) = (1, a - 1, \ell - a)$, which contradicts to the condition on a .

If $\ell \leq 6$, the condition for the lemma automatically holds since we have $\mu_P(\mathcal{A}) \geq 1$ for some $P \in \mathbb{P}_{\mathbb{C}}^2$ and $a \leq \lfloor (\ell - 1)/2 \rfloor \leq 2$. Therefore we have $\mathcal{S}_{\ell} = \emptyset$. \square

Now we introduce the invariant $F(\mathcal{A})$, which will be used to classify \mathcal{S}_{ℓ} .

Definition 2.12. Let \mathcal{A} be a line arrangement in $\mathbb{P}_{\mathbb{C}}^2$. We denote

$$M_i(\mathcal{A}) = \{P \in \mathbb{P}_{\mathbb{C}}^2 \mid \mu_P(\mathcal{A}) = i\}$$

and set the invariant $F(\mathcal{A})$ as

$$F(\mathcal{A}) = [F_1(\mathcal{A}), F_2(\mathcal{A}), \dots], \quad F_i(\mathcal{A}) = |M_i(\mathcal{A})| \quad (i = 1, 2, \dots).$$

Lemma 2.13. *The invariant $F(\mathcal{A})$ satisfies the following formulae.*

$$\sum_i i F_i(\mathcal{A}) = \mu_{\mathcal{A}}, \quad \sum_i (i+1) F_i(\mathcal{A}) = \sum_{H \in \mathcal{A}} n_{\mathcal{A}, H}, \quad \sum_i \binom{i+1}{2} F_i(\mathcal{A}) = \binom{|\mathcal{A}|}{2}.$$

Proof. The left equation is clear by definition. Since $P \in M_i(\mathcal{A})$ is contained in $(i+1)$ lines of \mathcal{A} , the middle equation holds. Finally, regarding all the intersection points of \mathcal{A} as the concentrations of the intersections of 2 lines of \mathcal{A} , we have the right equation. \square

Now we determine all the possibilities of $F(\mathcal{A})$ for $\mathcal{A} \in \mathcal{S}_{\ell}$ ($\ell \leq 12$).

Proposition 2.14. *Let $\ell \in \mathbb{Z}_{\leq 12}$ and $\mathcal{A} \in \mathcal{S}_{\ell}$ with $\exp(\mathcal{A}) = (1, a, b)$. Then we have*

$$(\ell, \min(a, b), F(\mathcal{A})) \in \left\{ \begin{array}{lll} (9, 4, [0, 12]), & (11, 5, [1, 14, 2]), & (11, 5, [4, 11, 3]), \\ (11, 5, [7, 8, 4]), & (11, 5, [10, 5, 5]), & (12, 5, [0, 16, 3]) \end{array} \right\}.$$

In particular, we have $\mathcal{S}_{\ell} = \emptyset$, $\mathcal{F}_{\ell} = \mathcal{I}_{\ell}$ for $2 \leq \ell \leq 8$ and $\mathcal{S}_{10} = \emptyset$.

Proof. Note that $b = \ell - 1 - a$. We may assume $a \leq (\ell - 1)/2$. By Lemma 2.11, we may assume $\ell \geq 7$ and $F_i = 0$ for $i \geq a - 1$. Since $\chi(\mathcal{A}, t) = (t-1)(t-a)(t-b)$ by Theorem 2.2, we have $\mu_{\mathcal{A}} = ab + \ell - 1 = (\ell - 1)(a + 1) - a^2$. Also, since $\mathcal{A} \in \mathcal{S}_{\ell}$, we have $\sum_{H \in \mathcal{A}} n_{\mathcal{A}, H} \leq a\ell$. Thus we have the inequalities as follows

$$\sum_{i=1}^{a-2} i F_i(\mathcal{A}) = (\ell - 1)(a + 1) - a^2, \quad \sum_{i=1}^{a-2} (i+1) F_i(\mathcal{A}) \leq a\ell, \quad \sum_{i=1}^{a-2} \binom{i+1}{2} F_i(\mathcal{A}) = \binom{\ell}{2}.$$

Solving above inequalities under the condition $0 \leq a \leq (\ell - 1)/2$ and $7 \leq \ell \leq 12$, we obtain only 6 triplets $[\ell, a, F]$ appearing in the right hand side of the statement. Now $\mathcal{S}_{\ell} = \emptyset$ for $2 \leq \ell \leq 8$ and $\mathcal{S}_{10} = \emptyset$ are clear. By Lemma 2.10, we have $\mathcal{F}_{\ell} = \mathcal{I}_{\ell}$ for $2 \leq \ell \leq 8$. \square

Definition 2.15. For $H \in \mathcal{A}$ and $i \in \mathbb{Z}_{>0}$, we set $\mu_{\mathcal{A},H} = \sum_{P \in H} \mu_P(\mathcal{A})$ and

$$M_i(H, \mathcal{A}) = M_i(\mathcal{A}) \cap H, \quad F_{H,i}(\mathcal{A}) = |M_i(H, \mathcal{A})|, \quad F_H(\mathcal{A}) = [F_{H,1}(\mathcal{A}), F_{H,2}(\mathcal{A}), \dots].$$

Lemma 2.16. For $H \in \mathcal{A}$, the invariant $F_H(\mathcal{A})$ satisfies the following formulae.

$$\sum_i F_{H,i}(\mathcal{A}) = n_{\mathcal{A},H}, \quad \sum_i i F_{H,i}(\mathcal{A}) = \mu_{\mathcal{A},H} = |\mathcal{A}| - 1, \quad \sum_{H \in \mathcal{A}} F_{H,i}(\mathcal{A}) = (i+1)F_i(\mathcal{A}).$$

Proof. The formulae above are clear by definitions and the fact that $|\mathcal{A}_P| = \mu_P(\mathcal{A}) + 1$. \square

In the following sections, we determine an arrangement $\mathcal{A} \in \mathcal{S}_\ell$ for $\ell \leq 12$. Namely, we determine the lattice structures of \mathcal{A} up to the permutations \mathfrak{S}_ℓ of indices of hyperplanes and determine their realizations in $\mathbb{P}_{\mathbb{C}}^2$ up to the action of $\text{PGL}(3, \mathbb{C})$.

The hyperplanes of \mathcal{A} are denoted by $\mathcal{A} = \{H_1, \dots, H_\ell\}$, while the defining equation of each H_i is denoted by h_i . The intersection points of \mathcal{A} satisfying $\mathcal{A}_P = \{H_{a_i} \mid i \in I\}$ is denoted by $\{a_i \mid i \in I\}$. The line passing through P and Q is denoted by \overline{PQ} . For the coordinate calculation, we regard $\mathbb{P}_{\mathbb{C}}^2$ as the union of affine part \mathbb{C}^2 and the infinity line H_∞ .

3. DETERMINATION OF \mathcal{S}_9

In this section, we show that \mathcal{S}_9 consists of dual Hesse arrangements.

3.1. Lattice structure of $\mathcal{A} \in \mathcal{S}_9$. We determine the lattice of $\mathcal{A} \in \mathcal{S}_9$. By Proposition 2.14, we have $F(\mathcal{A}) = [0, 12]$. Note that $F_H(\mathcal{A}) = [0, 4]$ for any $H \in \mathcal{A}$ since $F(\mathcal{A}) = [0, 12]$ and $\sum_i i F_{H,i}(\mathcal{A}) = \ell - 1 = 8$. Concerning $M_2(H_9, \mathcal{A})$, we may set

$$\{1, 2, 9\}, \{3, 4, 9\}, \{5, 6, 9\}, \{7, 8, 9\} \in M_2(\mathcal{A}).$$

Since $H_1 \cap H_3$ lies on H_5, H_6, H_7 or H_8 , we may set $\{1, 3, 5\} \in M_2(\mathcal{A})$ by symmetry. Since $H_1 \cap H_7 \neq H_1 \cap H_8$, they are other 2 points of $M_2(H_1, \mathcal{A})$. Thus we may set $\{1, 4, 7\}, \{1, 6, 8\} \in M_2(H_1, \mathcal{A})$ by symmetry of $(3, 5)(4, 6)$. Namely,

$$\{1, 3, 5\}, \{1, 4, 7\}, \{1, 6, 8\} \in M_2(\mathcal{A}).$$

Investigating $M_2(H_3, \mathcal{A})$, $M_2(H_4, \mathcal{A})$ and $M_2(H_2, \mathcal{A})$, it is easy to see

$$\{2, 3, 8\}, \{3, 6, 7\}, \{2, 4, 6\}, \{4, 5, 8\}, \{2, 5, 7\} \in M_2(\mathcal{A}).$$

Now we obtain all the points of $M_2(\mathcal{A})$, thus the lattice structure of \mathcal{A} is determined.

3.2. Realization of $\mathcal{A} \in \mathcal{S}_9$. We determine the realization of $\mathcal{A} \in \mathcal{S}_9$ in $\mathbb{P}_{\mathbb{C}}^2$. We may set H_9 as the infinity line H_∞ , $h_1 = x$, $h_2 = x - 1$, $h_3 = y$, $h_4 = y - 1$ and

$$\{1, 6, 8\} = (0, p), \{2, 5, 7\} = (1, q), \{3, 6, 7\} = (r, 0), \{4, 5, 8\} = (s, 1) \quad (p, q, r, s \neq 0, 1).$$

Note that $(0, 0), (1, q), (s, 1) \in H_5$, $(1, 1), (0, p), (r, 0) \in H_6$, $(0, 1), (1, q), (r, 0) \in H_7$ and $(1, 0), (0, p), (s, 1) \in H_8$. Therefore we have

$$sq = (1 - r)(1 - p) = r(1 - q) = p(1 - s) = 1.$$

Solving these equations, we have

$$(p, q, r, s) = (-\omega^2, -\omega, -\omega, -\omega^2),$$

where ω is a primitive third root of unity, and h_i for $5 \leq i \leq 8$ as follows.

$$h_5 = y + \omega x, \quad h_6 = y + \omega x + \omega^2, \quad h_7 = y - \omega^2 x - 1, \quad h_8 = y - \omega^2 x + \omega^2.$$

By this construction, for the permutation $\sigma \in \mathfrak{S}_9^* = \{\sigma \in \mathfrak{S}_9 \mid \sigma(L(\mathcal{A})) = L(\mathcal{A})\}$ preserving the lattice, there exists a $\text{GL}(3, \mathbb{C})$ -action sending each H_i to $H_{\sigma(i)}$, or sending each H_i to $\overline{H_{\sigma(i)}}$, where $\overline{H_i}$ stands for the Galois conjugate of H_i by $\text{Gal}(\mathbb{Q}[\sqrt{-3}]/\mathbb{Q})$. Note also that \mathcal{A} is transferred to $\overline{\mathcal{A}}$ by $[(x, y, z) \mapsto (y, x, z)] \in \text{GL}(3, \mathbb{C})$, which sends H_i to $\overline{H_{\mu(i)}}$ where $\mu = (1, 3)(2, 4)(7, 8) \in \mathfrak{S}_9^*$. Thus \mathcal{A} is realized uniquely up to the $\text{GL}(3, \mathbb{C})$ -action.

3.3. Verifying $\mathcal{A} \in \mathcal{S}_9 \subset \mathcal{R}_9$. We check the freeness of \mathcal{A} realized in §3.2 and show that $\mathcal{A} \in \mathcal{R}_9$. We set $\mathcal{A}_1 = \mathcal{A} \cup \{H_{10}\}$ where $h_{10} = x - y$. Then we have

$$\mathcal{A}_1 \cap H_{10} = \{(0, 0), (1, 1), ((1 - \omega^2)^{-1}, (1 - \omega^2)^{-1}), ((1 - \omega)^{-1}, (1 - \omega)^{-1}), H_{10} \cap H_\infty\}.$$

Since $\mu_{\mathcal{A}_1} = \mu_{\mathcal{A}} + 5$, we have $\chi(\mathcal{A}_1, t) = (t - 1)(t - 4)(t - 5)$. By Theorem 2.8, we have $\mathcal{A}_1 \in \mathcal{F}_{10}$ with $\exp(\mathcal{A}_1) = (1, 4, 5)$, and hence $\mathcal{A} \in \mathcal{S}_9$. We set $\mathcal{A}_2 = \mathcal{A}_1 \setminus \{H_\infty\}$ and $\mathcal{A}_3 = \mathcal{A}_2 \setminus \{H_7\}$. Since $n_{\mathcal{A}_1, H_\infty} = 5$, we have $\mathcal{A}_2 \in \mathcal{F}_9$ with $\exp(\mathcal{A}_2) = (1, 4, 4)$. Since $n_{\mathcal{A}_2, H_7} = 5$, we have $\mathcal{A}_3 \in \mathcal{F}_8 = \mathcal{I}_8$. Therefore $\mathcal{A}_2 \in \mathcal{I}_9$, $\mathcal{A}_1 \in \mathcal{I}_{10}$ and $\mathcal{A} \in \mathcal{R}_9$.

In fact, to check whether $\mathcal{A} \in \mathcal{F}_9$ belongs to \mathcal{S}_9 or not, we have only to check $F(\mathcal{A})$.

Lemma 3.1. *If $\mathcal{A} \in \mathcal{F}_9$ satisfies $F(\mathcal{A}) = [0, 12]$, then $\mathcal{A} \in \mathcal{S}_9$.*

Proof. For any $H \in \mathcal{A}$, since $9 - 1 = \mu_{\mathcal{A}, H} = 2n_{\mathcal{A}, H}$, we have $n_{\mathcal{A}, H} = 4$. □

Definition 3.2. An arrangement in \mathbb{C}^3 is called a *dual Hesse* arrangement if it is $\text{GL}(3, \mathbb{C})$ -equivalent to $(\varphi_{\text{dH}} = 0)$, where

$$\varphi_{\text{dH}} = (x^3 - y^3)(y^3 - z^3)(z^3 - x^3).$$

It is easy to see that $\mathcal{A} = (\varphi_{\text{dH}} = 0)$ satisfies $F(\mathcal{A}) = [0, 12]$. Therefore,

$$\mathcal{S}_9 = \{\text{dual Hesse arrangements}\} \subset \mathcal{R}_9.$$

3.4. Addition to $\mathcal{A} \in \mathcal{S}_9$. The structures of \mathcal{F}_9 and \mathcal{F}_{10} are given as below.

Proposition 3.3.

(1) *Let $H \in \mathcal{A}_1 \in \mathcal{F}_{10}$ such that $\mathcal{A} = \mathcal{A}_1 \setminus \{H\} \in \mathcal{S}_9$. Then, $\mathcal{A}_1 \in \mathcal{I}_{10}$ and \mathcal{A}_1 is unique up to the $\text{GL}(3, \mathbb{C})$ -action.*

(2) $\mathcal{F}_9 = \mathcal{R}_9 = \mathcal{I}_9 \sqcup \mathcal{S}_9$ and $\mathcal{F}_{10} = \mathcal{I}_{10}$.

Proof.

(1) We may assume \mathcal{A} has the description as in §3.1 and §3.2. By Theorem 2.4, we have $n_{\mathcal{A}_1, H} = 5$ and hence $F_H(\mathcal{A}_1) = [3, 0, 2]$. Since $H \notin \mathcal{A}$, H is one of the following

$$\begin{aligned} & \overline{\{1, 2, 9\}\{3, 6, 7\}}, \overline{\{1, 2, 9\}\{4, 5, 8\}}, \overline{\{1, 3, 5\}\{2, 4, 6\}}, \overline{\{1, 3, 5\}\{7, 8, 9\}}, \\ & \overline{\{1, 4, 7\}\{2, 3, 8\}}, \overline{\{1, 4, 7\}\{5, 6, 9\}}, \overline{\{1, 6, 8\}\{2, 5, 7\}}, \overline{\{1, 6, 8\}\{3, 4, 9\}}, \\ & \overline{\{2, 3, 8\}\{5, 6, 9\}}, \overline{\{2, 4, 6\}\{7, 8, 9\}}, \overline{\{2, 5, 7\}\{3, 4, 9\}}, \overline{\{3, 6, 7\}\{4, 5, 8\}}. \end{aligned}$$

Recall that any $\sigma \in \mathfrak{S}_9^*$ is realized by the action of $\text{GL}(3, \mathbb{C})$ and $\text{Gal}(\mathbb{Q}[\sqrt{-3}]/\mathbb{Q})$. Therefore it suffices to show that \mathfrak{S}_9^* acts transitively on the pairs in the above list.

Observe that each points of $M_2(\mathcal{A})$ lies on two candidates of H . We denote the \mathfrak{S}_9^* -equivalence by the symbol “ \sim ”. First note that $(\{1, 2, 9\}, \{3, 6, 7\}) \sim (\{3, 6, 7\}, \{1, 2, 9\})$ by $(1, 3)(2, 6)(7, 9) \in \mathfrak{S}_9^*$ and $(\{1, 2, 9\}, \{3, 6, 7\}) \sim (\{1, 2, 9\}, \{4, 5, 8\})$ by $(3, 4)(5, 7)(6, 8) \in \mathfrak{S}_9^*$. As the point transferred from $\{1, 2, 9\}$ has the same property as above, it follows that $(\{1, 2, 9\}, \{3, 6, 7\}) \sim (\{3, 6, 7\}, \{4, 5, 8\})$. Namely, we have

$$(\{1, 2, 9\}, \{3, 6, 7\}) \sim (\{1, 2, 9\}, \{4, 5, 8\}) \sim (\{3, 6, 7\}, \{4, 5, 8\}).$$

By applying $(2, 3)(4, 7)(5, 9), (2, 7)(4, 9)(6, 8), (2, 6)(3, 5)(8, 9) \in \mathfrak{S}_9^*$, We have

$$(\{1, 2, 9\}, \{3, 6, 7\}) \sim (\{1, 3, 5\}, \{2, 4, 6\}) \sim (\{1, 4, 7\}, \{2, 3, 8\}) \sim (\{1, 6, 8\}, \{2, 5, 7\}).$$

Therefore we have the following, which completes the proof of the uniqueness of \mathcal{A}_1 .

$$\begin{aligned} (\{1, 2, 9\}, \{3, 6, 7\}) & \sim (\{1, 3, 5\}, \{2, 4, 6\}) \sim (\{1, 3, 5\}, \{7, 8, 9\}) \sim (\{2, 4, 6\}, \{7, 8, 9\}) \\ & \sim (\{1, 4, 7\}, \{2, 3, 8\}) \sim (\{1, 4, 7\}, \{5, 6, 9\}) \sim (\{2, 3, 8\}, \{5, 6, 9\}) \\ & \sim (\{1, 6, 8\}, \{2, 5, 7\}) \sim (\{1, 6, 8\}, \{3, 4, 9\}) \sim (\{2, 5, 7\}, \{3, 4, 9\}). \end{aligned}$$

Note that H_{10} in §3.3 is $\overline{\{1, 3, 5\}\{2, 4, 6\}}$ and $\mathcal{A} \cup \{H_{10}\} \in \mathcal{I}_{10}$. By the uniqueness of \mathcal{A}_1 , we conclude that $\mathcal{A}_1 \in \mathcal{I}_{10}$. Therefore (1) is verified.

(2) Since $\mathcal{F}_8 = \mathcal{I}_8$ and $\mathcal{S}_9 \subset \mathcal{R}_9$, we have $\mathcal{F}_9 = \mathcal{I}_9 \sqcup \mathcal{S}_9 = \mathcal{R}_9$ by Lemma 2.10. Let $\mathcal{A} \in \mathcal{F}_{10}$. Since $\mathcal{S}_{10} = \emptyset$, there exists $H \in \mathcal{A}$ such that $\mathcal{A}' = \mathcal{A} \setminus \{H\} \in \mathcal{F}_9 = \mathcal{I}_9 \sqcup \mathcal{S}_9$. If $\mathcal{A}' \in \mathcal{I}_9$, then $\mathcal{A} \in \mathcal{I}_{10}$. If $\mathcal{A}' \in \mathcal{S}_9$, we also have $\mathcal{A} \in \mathcal{I}_{10}$ by (1). Therefore we have $\mathcal{F}_{10} = \mathcal{I}_{10}$. \square

We remark that now Theorem 1.1 is established for $|\mathcal{A}| \leq 10$ by Propositions 2.14 and 3.3. We give the proof of Corollary 1.3.

Proof of Corollary 1.3. The proof is by the induction on $\ell = |\mathcal{A}|$. If $\ell \leq 10$, we have nothing to prove. Assume that $\ell \geq 11$. If $\mathcal{A} \in \mathcal{F}_\ell \setminus \mathcal{S}_\ell$, then $H \in \mathcal{A}$ such that $\mathcal{A}' = \mathcal{A} \setminus \{H\} \in \mathcal{F}_{\ell-1}$. Since $\exp(\mathcal{A}') = (1, a-1, b)$ or $(1, a, b-1)$, we have $\mathcal{A}' \in \mathcal{I}_{\ell-1}$ by induction hypothesis, and hence $\mathcal{A} \in \mathcal{I}_\ell$. Thus we may assume $\mathcal{A} \in \mathcal{S}_\ell$. We set $a \leq b$ and take $H \in \mathcal{A}$. By Lemma 2.11, we have $\mu_P(\mathcal{A}) \leq a-2$ for any $P \in H$. By definition of \mathcal{S}_ℓ , we have $n_{\mathcal{A},H} \leq a$. However it is a contradiction since we have the following inequalities.

$$11 - 1 \leq \ell - 1 = \mu_{\mathcal{A},H} \leq (a-2)a \leq (4-2) \cdot 4 = 8. \quad \square$$

4. DETERMINATION OF \mathcal{S}_{11}

In this section, we show that \mathcal{S}_{11} consists of pentagonal arrangements.

4.1. Absence of $\mathcal{A} \in \mathcal{S}_{11}$ with $F(\mathcal{A}) = [1, 14, 2]$. Let $\mathcal{A} \in \mathcal{S}_{11}$. By Proposition 2.14, we have $F(\mathcal{A}) = [1, 14, 2], [4, 11, 3], [7, 8, 4]$ or $[10, 5, 5]$.

Suppose $F(\mathcal{A}) = [1, 14, 2]$. Take $P \in M_3(\mathcal{A})$. Since $|M_1(\mathcal{A}) \cup M_3(\mathcal{A}) \setminus \{P\}| = 2$ and $|\mathcal{A}_P| = 4$, there exists $H \in \mathcal{A}_P$ such that $M_1(H, \mathcal{A}) = \emptyset$ and $M_3(H, \mathcal{A}) = \{P\}$. Then it follows that $11 - 1 = \mu_{\mathcal{A},H} = 0 + 2F_{H,2}(\mathcal{A}) + 3 \cdot 1$, which is impossible.

Therefore $F(\mathcal{A}) = [4, 11, 3], [7, 8, 4]$ or $[10, 5, 5]$. In the following subsections, we show that only the case $F(\mathcal{A}) = [10, 5, 5]$ occurs, which corresponds to the case when \mathcal{A} is a pentagonal arrangement.

4.2. Subarrangement \mathcal{A}' of \mathcal{A} . In the case $F(\mathcal{A}) = [4, 11, 3]$ or $[7, 8, 4]$, we construct a subarrangement $\mathcal{A}' = \{H_1, \dots, H_{10}\}$ of \mathcal{A} satisfying the following.

$$F(\mathcal{A}') = [9, 6, 3], \quad n_{\mathcal{A}', H_i} = \begin{cases} 4 & i = 1 \\ \leq 5 & 2 \leq i \leq 10 \end{cases}, \quad M_3(\mathcal{A}') = \begin{cases} H_1 \cap H_2, \\ H_1 \cap H_3, \\ H_2 \cap H_3 \end{cases}. \quad (*)$$

Suppose $F(\mathcal{A}) = [4, 11, 3]$. Note that $n_{\mathcal{A},H} = 4, 5$ for any $H \in \mathcal{A}$, since $3 \cdot 3 < 10 = \mu_{\mathcal{A},H}$. Thus $n_{\mathcal{A},H} = 4, 5$ for any $H \in \mathcal{A}$. Since $\sum_{H \in \mathcal{A}} n_{\mathcal{A},H} = 2 \cdot 4 + 3 \cdot 11 + 4 \cdot 3 = 5 \cdot 11 - 2$, we may set $n_{\mathcal{A},H_1} = n_{\mathcal{A},H_2} = 4$ and $n_{\mathcal{A},H_i} = 5$ for $3 \leq i \leq 11$. Note that $F_{H_i,3}(\mathcal{A}) \geq 2$ for $i = 1, 2$, since $2 \cdot 3 + 3 \cdot 1 < 10 = \mu_{\mathcal{A},H}$. Thus we may set $M_3(\mathcal{A}) = \{P_1, P_2, P_3\}$, $H_1 = \overline{P_2 P_3}$ and $H_2 = \overline{P_1 P_3}$. Since $n_{\mathcal{A},H_1} = 4 = |\mathcal{A}_{P_1} \cap H_1|$, we have $\overline{P_1 P_2} \in \mathcal{A}$, which we set H_3 . Since $|\bigcup_{P \in M_3(\mathcal{A})} \mathcal{A}_P| = 4 \cdot 3 - 3 = 9 < 11$, we may set $H_{11} \cap M_3(\mathcal{A}) = \emptyset$. Then $F_{\mathcal{A},H_{11}} = [0, 5, 0]$. Now it is easy to check that $\mathcal{A}' = \mathcal{A} \setminus \{H_{11}\}$ satisfies the condition (*).

Suppose $F(\mathcal{A}) = [7, 8, 4]$. Since $\sum_{H \in \mathcal{A}} n_{\mathcal{A},H} = 2 \cdot 7 + 3 \cdot 8 + 4 \cdot 4 = 5 \cdot 11 - 1$, we may assume $n_{\mathcal{A},H_1} = 4$ and $n_{\mathcal{A},H_i} = 5$ for $2 \leq i \leq 11$. Note that $F_{H_1,3}(\mathcal{A}) \neq 1, 4$ since $2 \cdot 3 + 3 \cdot 1 < \mu_{\mathcal{A},H_1} = 10 < 3 \cdot 4$. Thus $F_{H_1,3}(\mathcal{A}) = 2, 3$. We set $M_3(\mathcal{A}) = \{P_1, P_2, P_3, P_4\}$ so that $P_1 \notin H_1 = \overline{P_2 P_3}$. Since $|\mathcal{A}_{P_1} \cap H_1| = 4 = n_{\mathcal{A},H_1}$, we have $P_2, P_3 \in \mathcal{A}_{P_1}$. Therefore we may set $H_2 = \overline{P_1 P_3}$ and $H_3 = \overline{P_1 P_2}$. Since $|\mathcal{A}_{P_4}| = 4$, we may set $M_3(\mathcal{A}) \cap H_{11} = \{P_4\}$. Since $n_{\mathcal{A},H_{11}} = 5$ and $F_{H_{11},3}(\mathcal{A}) = 1$, we have $F_{H_{11}}(\mathcal{A}) = [1, 3, 1]$. Now it is easy to check that $\mathcal{A}' = \mathcal{A} \setminus \{H_{11}\}$ satisfies the condition (*).

4.3. Lattice structure of \mathcal{A}' . First we determine $F_{H_i}(\mathcal{A}')$ for $1 \leq i \leq 10$. We have $n_{\mathcal{A}',H} = 4, 5$ for any $H \in \mathcal{A}'$ since $|M_3(H, \mathcal{A}')| \leq 2$ and $2+3 \cdot 2 < 9 = \mu_{\mathcal{A}',H}$. Since $n_{\mathcal{A}',H_1} = 4$ and $F_{3,H_1}(\mathcal{A}') = 2$, we have $F_{H_1}(\mathcal{A}') = [1, 1, 2]$. Since $\bigcup_{P \in M_3(\mathcal{A}')} \mathcal{A}'_P = 4 \cdot 3 - 3 = 9$, we may set $M_3(H_{10}, \mathcal{A}') = \emptyset$, and hence $F_{H_{10}} = [1, 4, 0]$. Since $H_2 \cap H_{10} \neq H_3 \cap H_{10}$, we may set $H_2 \cap H_{10} \in M_2(\mathcal{A}')$. Since $F_{H_2,3}(\mathcal{A}') = 2$, we have $F_{H_2}(\mathcal{A}') = [1, 1, 2]$. Since $\sum_{H \in \mathcal{A}'} n_{\mathcal{A}',H} = 2 \cdot 9 + 3 \cdot 6 + 4 \cdot 3 = 5 \cdot 10 - 2$, we have $n_{\mathcal{A}',H_i} = 4$ for $i = 1, 2$ and $n_{\mathcal{A}',H_i} = 5$

for $3 \leq i \leq 10$. We have $F_{H_3}(\mathcal{A}') = [3, 0, 2]$ and $F_{H_i}(\mathcal{A}') = [2, 2, 1]$ for $4 \leq i \leq 9$ since $F_{H_3,3}(\mathcal{A}') = 2$ and $F_{H_i,3}(\mathcal{A}') = 1$. As a conclusion, we have the following.

$$F_{H_i}(\mathcal{A}') = [1, 1, 2] \ (i = 1, 2), [3, 0, 2] \ (i = 3), [2, 2, 1] \ (4 \leq i \leq 9), [1, 4, 0] \ (i = 10).$$

Now we determine the lattice structure of \mathcal{A}' . We may set

$$M_3(\mathcal{A}') = \{P_1 = \{2, 3, 4, 5\}, P_2 = \{1, 3, 6, 7\}, P_3 = \{1, 2, 8, 9\}\}.$$

Note that $\{3, 8\}, \{3, 9\}, \{3, 10\} \in M_1(\mathcal{A}')$ since $F_{H_3}(\mathcal{A}') = [3, 0, 2]$. By symmetry of $(4, 5)$ or $(6, 7)$, we may set $\{1, 4, 10\}, \{2, 6, 10\} \in M_2(\mathcal{A}')$. Since $H_8 \cap H_{10}$ lies on H_5 or H_7 , we may set $\{5, 8, 10\} \in M_2(\mathcal{A}')$ by symmetry of $(1, 2)(4, 6)(5, 7)$. We also have $\{7, 9, 10\} \in M_2(\mathcal{A}')$. Since $\{\{7, 9, 10\} \cup (\mathcal{A}'_{P_1} \cap H_7)\}$ defines all intersection points on H_7 , we have $H_7 \cap H_8 \in M_2(\mathcal{A}')$. By the same reasoning for $\{\{5, 8, 10\} \cup (\mathcal{A}'_{P_2} \cap H_8)\}$ on H_8 , we have $H_4 \cap H_8 \in M_2(\mathcal{A}')$. Thus we have $M_2(H_8, \mathcal{A}') = \{\{5, 8, 10\}, \{4, 7, 8\}\}$. Since $F_{H_i,2}(\mathcal{A}') = 2$ for $i = 5, 6, 9$, The last point of $M_2(\mathcal{A}')$ is $\{5, 6, 9\}$. Therefore $M_2(\mathcal{A}')$, and hence $M_1(\mathcal{A}')$, are as follows, which determine the lattice of \mathcal{A}' .

$$\begin{aligned} M_2(\mathcal{A}') &= \{\{1, 4, 10\}, \{2, 6, 10\}, \{4, 7, 8\}, \{5, 6, 9\}, \{5, 8, 10\}, \{7, 9, 10\}\}, \\ M_1(\mathcal{A}') &= \{\{1, 5\}, \{2, 7\}, \{3, 8\}, \{3, 9\}, \{3, 10\}, \{4, 6\}, \{4, 9\}, \{5, 7\}, \{6, 8\}\}. \end{aligned}$$

4.4. Realization of \mathcal{A}' . We determine the realization of \mathcal{A}' in $\mathbb{P}_{\mathbb{C}}^2$. We may set H_{10} as the infinity line H_{∞} , $P_1 = (1, 0)$, $P_2 = (0, 1)$ and $P_3 = (0, 0)$. Then

$$h_1 = x, \ h_2 = y, \ h_3 = x + y - 1, \ h_4 = x - 1, \ h_6 = y - 1.$$

Set $\{4, 7, 8\} = (1, p)$ and $\{5, 6, 9\} = (q, 1)$ ($p, q \neq 0$). Then we have

$$h_5 = x - (q - 1)y - 1, \ h_7 = (p - 1)x - y + 1, \ h_8 = px - y, \ h_9 = x - qy.$$

Since $H_5 \parallel H_8$ and $H_7 \parallel H_9$, we have $p(q - 1) = (p - 1)q = 1$. Therefore we conclude that $p = q = \zeta$ where ζ is a solution of $\zeta^2 - \zeta - 1 = 0$, and we may reset the equations as

$$h_5 = \zeta x - y - \zeta, \ h_7 = x - \zeta y + \zeta, \ h_8 = \zeta x - y, \ h_9 = x - \zeta y.$$

4.5. Absence of $\mathcal{A} \in \mathcal{S}_{11}$ with $F(\mathcal{A}) = [4, 11, 3]$ or $[7, 8, 4]$. We show that we cannot extend the realization of \mathcal{A}' obtained above to \mathcal{A} . Assume that $\mathcal{A} = \mathcal{A}' \cup \{H_{11}\}$ is realizable.

Suppose $F(\mathcal{A}) = [4, 11, 3]$. Recall that $F_{H_{11}}(\mathcal{A}) = [0, 5, 0]$. Since $|M_1(\mathcal{A}') \cap H_{11}| = 5$ and $M_1(\mathcal{A}') \subset \{\{1, 5\}, \{4, 9\}\} \cup \bigcup_{i=3,6,7} H_i$, we have $H_{11} = \overline{\{1, 5\}\{4, 9\}} = \overline{(0, -\zeta)(1, \zeta^{-1})}$ and $h_{11} = (1 - 2\zeta)x + y + \zeta$. Therefore $H_{10} \cap H_{11} \in M_1(\mathcal{A})$, a contradiction.

Suppose $F(\mathcal{A}) = [7, 8, 4]$. Recall that $F_{H_{11}}(\mathcal{A}) = [1, 3, 1]$. Thus $M_1(H_{11}, \mathcal{A}) = \{H_i \cap H_{11}\}$ for some $1 \leq i \leq 10$. Since $n_{\mathcal{A}', H_i} + 1 = n_{\mathcal{A}, H_i} \leq 5$, we have $i = 1$ or 2 . We may set $H_1 \cap H_{11} \in M_1(\mathcal{A})$ by the symmetry of the coordinates x and y . Note that $|M_1(\mathcal{A}') \cap H_{11}| = 3$ and $|M_2(\mathcal{A}') \cap H_{11}| = 1$. In particular, $H_2 \cap H_{11} = \{2, 7\}$ or $\{2, 6, 10\}$.

Assume that $H_2 \cap H_{11} = \{2, 7\} = (-\zeta, 0)$. Then $M_2(\mathcal{A}') \cap H_{11} = \{\{5, 6, 9\}\}$ or $\{\{5, 8, 10\}\}$. If $H_{11} \ni \{5, 6, 9\} = (\zeta, 1)$, we have $h_{11} = x - 2\zeta y + \zeta$. Therefore $H_{10} \cap H_{11} \in M_1(\mathcal{A})$, a contradiction. If $H_{11} \ni \{5, 8, 10\}$, we have $H_{11} \cap M_1(\mathcal{A}') = \{\{2, 7\}, \{3, 9\}, \{4, 6\}\}$. Since $\{4, 6\} = (1, 1)$, we have $h_{11} = x - (\zeta + 1)y + \zeta$, which contradicts to $H_{11} \parallel H_8$.

Assume that $H_2 \cap H_{11} = \{2, 6, 10\}$. Then we have $M_1(\mathcal{A}') \cap H_{11} = \{\{3, 8\}, \{4, 9\}, \{5, 7\}\}$. Since $\{4, 9\} = (1, \zeta - 1)$ and $\{5, 7\} = (\zeta + 1, \zeta + 1)$ we have $H_{11} \nparallel H_2$, a contradiction.

Now we may assume that $F(\mathcal{A}) = [10, 5, 5]$.

4.6. Lattice structure of $\mathcal{A} \in \mathcal{S}_{11}$. We determine the lattice of $\mathcal{A} \in \mathcal{S}_{11}$. First we show that $\overline{PQ} \in \mathcal{A}$ for any $P, Q \in M_3(\mathcal{A})$, $P \neq Q$. Assume that there exist $P, Q \in M_3(\mathcal{A})$ such that $\overline{PQ} \notin \mathcal{A}$. Note that \mathcal{A} has $10 + 5 + 5 = 20$ intersection points, and $\mathcal{A}_P \cup \mathcal{A}_Q$ covers $4 \cdot 4 + 2 = 18$ of them. We set the left 2 intersection points in $\mathcal{A} \setminus (\mathcal{A}_P \cup \mathcal{A}_Q)$ as T_1 and T_2 . If $H = \overline{T_1 T_2} \in \mathcal{A}$, then $\{T_1, T_2\} \cap (\mathcal{A}_P \cap H) \neq \emptyset$ since $|\mathcal{A}_P \cap H| = 4$ and $n_{\mathcal{A}, H} \leq 5$. However, it contradicts to the choice of T_i . If $\overline{T_1 T_2} \notin \mathcal{A}$, then $(\mathcal{A}_P \cup \mathcal{A}_Q) \cap (\mathcal{A}_{T_1} \cup \mathcal{A}_{T_2}) \neq \emptyset$ since $|\mathcal{A}| = 11$, $|\mathcal{A}_P \cup \mathcal{A}_Q| = 8$ and $|\mathcal{A}_{T_1} \cup \mathcal{A}_{T_2}| \geq 4$. It also contradicts to the choice of T_i .

Next we determine $F_{H_i}(\mathcal{A})$ for $1 \leq i \leq 10$. Note that $n_{\mathcal{A}, H} = 5$ for $H \in \mathcal{A}$ since $\sum_{H \in \mathcal{A}} n_{\mathcal{A}, H} = 2 \cdot 10 + 3 \cdot 5 + 4 \cdot 5 = 5 \cdot 11$. We also have $F_{H,3}(\mathcal{A}) \leq 2$ for $H \in \mathcal{A}$ since $\mu_{\mathcal{A}, H} = 10 < 1 \cdot 2 + 3 \cdot 3$. It follows that, for $P, Q \in M_3(\mathcal{A})$ with $P \neq Q$, $\overline{PQ} \in \mathcal{A}$ are distinct each other, forming $\binom{5}{2} = 10$ lines of \mathcal{A} . Thus we may assume $F_{H,3}(\mathcal{A}) = 2$, i.e., $F_{H_i}(\mathcal{A}) = [2, 1, 2]$, for $1 \leq i \leq 10$. Since $4 \cdot 5 = \sum_{H \in \mathcal{A}} F_{H,3}(\mathcal{A}) = 2 \cdot 10 + F_{H_{11},3}(\mathcal{A})$, we have $F_{H_{11},3}(\mathcal{A}) = 0$, i.e., $F_{H_{11}}(\mathcal{A}) = [0, 5, 0]$. Therefore, we have

$$F_{H_i}(\mathcal{A}) = [2, 1, 2] \ (1 \leq i \leq 10), \quad F_{H_{11}}(\mathcal{A}) = [0, 5, 0].$$

We investigate the lattice structure of \mathcal{A} . We may set $M_3(\mathcal{A}) = \{P_i \mid 1 \leq i \leq 5\}$ and

$$\begin{aligned} H_1 &= \overline{P_1 P_2}, \ H_2 = \overline{P_1 P_3}, \ H_3 = \overline{P_1 P_4}, \ H_4 = \overline{P_1 P_5}, \ H_5 = \overline{P_2 P_3}, \\ H_6 &= \overline{P_2 P_4}, \ H_7 = \overline{P_2 P_5}, \ H_8 = \overline{P_3 P_4}, \ H_9 = \overline{P_3 P_5}, \ H_{10} = \overline{P_4 P_5}, \end{aligned}$$

or, in other words, $M_3(\mathcal{A})$ consists of the following five points.

$$P_1 = \{1, 2, 3, 4\}, \ P_2 = \{1, 5, 6, 7\}, \ P_3 = \{2, 5, 8, 9\}, \ P_4 = \{3, 6, 8, 10\}, \ P_5 = \{4, 7, 9, 10\}.$$

Since $H_1 \cap H_{11} \in M_2(\mathcal{A})$ lies on H_8, H_9 or H_{10} , we may set $\{1, 9, 11\} \in M_2(\mathcal{A})$ by symmetry. Since $H_3 \cap H_{11} \in M_2(\mathcal{A})$ lies on H_5 or H_7 , we may set $\{3, 5, 11\} \in M_2(\mathcal{A})$ by symmetry of $(2, 4)(5, 7)(8, 10)$. Investigating $H_{10} \cap H_{11}, H_6 \cap H_{11}, H_8 \cap H_{11} \in M_2(\mathcal{A})$ in this order, we have $\{2, 10, 11\}, \{4, 6, 11\}, \{7, 8, 11\} \in M_2(\mathcal{A})$. Thus $M_2(\mathcal{A})$ is determined.

$$M_2(\mathcal{A}) = \{\{1, 9, 11\}, \{2, 10, 11\}, \{3, 5, 11\}, \{4, 6, 11\}, \{7, 8, 11\}\}.$$

Now $M_2(\mathcal{A})$ and $M_3(\mathcal{A})$ are determined, which gives the lattice structure of \mathcal{A} .

4.7. Realization of $\mathcal{A} \in \mathcal{S}_{11}$. We determine the realization of $\mathcal{A} \in \mathcal{S}_{11}$ in $\mathbb{P}_{\mathbb{C}}^2$. We may set H_{11} as the infinity line H_{∞} , $P_1 = (0, 1)$, $P_2 = (0, 0)$ and $P_3 = (1, 0)$. By definition of H_1, H_2, H_5 and the fact that $P_3 \in H_9 \parallel H_1$ and $P_1 \in H_3 \parallel H_5$ imply that

$$h_1 = x, \ h_2 = x + y - 1, \ h_3 = y - 1, \ h_5 = y, \ h_9 = x - 1.$$

We set $P_4 = (p, 1)$ and $P_5 = (1, q)$. Since $H_{10} = \overline{P_4 P_5} \parallel H_2$ and $H_4 = \overline{P_1 P_5} \parallel H_6 = \overline{P_2 P_4}$, we have $p = q$ and $p(q - 1) = 1$. Thus we have $p = q = \zeta$ where ζ is a solution of $\zeta^2 - \zeta - 1 = 0$. The left defining equations h_i of H_i are as follows.

$$h_4 = x - \zeta y + \zeta, \ h_6 = x - \zeta y, \ h_7 = \zeta x - y, \ h_8 = \zeta x - y - \zeta, \ h_{10} = x + y - \zeta - 1.$$

By this construction, for the permutation $\sigma \in \mathfrak{S}_{11}^* = \{\sigma \in \mathfrak{S}_{11} \mid \sigma(L(\mathcal{A})) = L(\mathcal{A})\}$, there exists a $\text{GL}(3, \mathbb{C})$ -action sending each H_i to $H_{\sigma(i)}$, or sending each H_i to $\overline{H_{\sigma(i)}}$, where $\overline{H_i}$ stands for the Galois conjugate of H_i by $\text{Gal}(\mathbb{Q}[\sqrt{5}]/\mathbb{Q})$. Note also that \mathcal{A} is transferred to $\overline{\mathcal{A}}$ by $[(x, y, z) \mapsto (\zeta x + y, x + \zeta y, z)] \in \text{GL}(3, \mathbb{C})$, which sends H_i to $\overline{H_{\nu(i)}}$ where $\nu = (1, 6, 5, 7)(2, 10)(3, 8, 9, 4) \in \mathfrak{S}_{11}^*$. Thus \mathcal{A} is realized uniquely up to the $\text{GL}(3, \mathbb{C})$ -action.

4.8. Verifying $\mathcal{A} \in \mathcal{S}_{11} \subset \mathcal{R}_{11}$. We check the freeness of \mathcal{A} realized in §4.7 and show that $\mathcal{A} \in \mathcal{R}_{11}$. We set $\mathcal{A}_1 = \mathcal{A} \cup \{H_{12}\}$ where $h_{12} = x - y$. Then we have

$$\mathcal{A}_1 \cap H_{12} = \left\{ (0, 0), (1, 1), \left(\frac{1}{2}, \frac{1}{2}\right), (\zeta + 1, \zeta + 1), \left(\frac{\zeta + 1}{2}, \frac{\zeta + 1}{2}\right), H_{12} \cap H_{\infty} \right\}.$$

Since $\mu_{\mathcal{A}_1} = \mu_{\mathcal{A}} + 6$, we have $\chi(\mathcal{A}_1, t) = (t - 1)(t - 5)(t - 6)$. Thus $\mathcal{A}_1 \in \mathcal{F}_{12}$ with $\exp(\mathcal{A}_1) = (1, 5, 6)$ by Theorem 2.8, and hence $\mathcal{A} \in \mathcal{S}_{11}$. We set $\mathcal{A}_2 = \mathcal{A}_1 \setminus \{H_{\infty}\}$ and

$\mathcal{A}_3 = \mathcal{A}_2 \setminus \{H_2\}$. Since $n_{\mathcal{A}_1, H_\infty} = 6$, we have $\mathcal{A}_2 \in \mathcal{F}_{11}$ with $\exp(\mathcal{A}_2) = (1, 5, 5)$. Since $n_{\mathcal{A}_2, H_2} = 6$, we have $\mathcal{A}_3 \in \mathcal{F}_{10} = \mathcal{I}_{10}$. Therefore, $\mathcal{A}_2 \in \mathcal{I}_{11}$, $\mathcal{A}_1 \in \mathcal{I}_{12}$ and $\mathcal{A} \in \mathcal{R}_{11}$.

Remark 4.1. Note that $\mathcal{A} \in \mathcal{F}_{11}$ satisfying $F(\mathcal{A}) = [10, 5, 5]$ does not necessary belong to \mathcal{S}_{11} . For example, the arrangement \mathcal{A} defined by the equation $xyz(x^2 - z^2)(y^2 - z^2)(x^2 - y^2)(x - y + z)(x - y + 2z)$ satisfies $F(\mathcal{A}) = [10, 5, 5]$ but $\mathcal{A} \in \mathcal{I}_{11}$.

Definition 4.2. The arrangement of Example 4.59 in [OT] is the cone of the line arrangement consisted of 5 sides and 5 diagonals of a regular pentagon, defined by the equation

$$\begin{aligned} \varphi_{\text{pen}} &= z(4x^2 + 2x - z) \\ &\quad (x^4 - 10x^2y^2 + 5y^4 + 6x^3z - 10xy^2z + 11x^2z^2 - 5y^2z^2 + 6xz^3 + z^4) \\ &\quad (x^4 - 10x^2y^2 + 5y^4 - 4x^3z + 20xy^2z + 6x^2z^2 - 10y^2z^2 - 4xz^3 + z^4). \end{aligned}$$

An arrangement in \mathbb{C}^3 is called *pentagonal* if it is $\text{GL}(3, \mathbb{C})$ -equivalent to $(\varphi_{\text{pen}} = 0)$.

It is easy to see that $\mathcal{A} = (\varphi_{\text{pen}} = 0)$ satisfies $\mathcal{A} \in \mathcal{S}_{11}$ and $F(\mathcal{A}) = [10, 5, 5]$. Therefore,

$$\mathcal{S}_{11} = \{\text{pentagonal arrangements}\} \subset \mathcal{R}_{11}.$$

By the description in §4.7, the lattice of a pentagonal arrangement is realized over $\mathbb{Q}[\sqrt{5}]$.

4.9. Addition to $\mathcal{A} \in \mathcal{S}_{11}$. The structures of \mathcal{F}_{11} and \mathcal{F}_{12} are given as below.

Proposition 4.3.

(1) Let $H \in \mathcal{A}_1 \in \mathcal{F}_{12}$ such that $\mathcal{A} = \mathcal{A}_1 \setminus \{H\} \in \mathcal{S}_{11}$. Then, $\mathcal{A}_1 \in \mathcal{I}_{12}$ and \mathcal{A}_1 has two possibilities up to the $\text{GL}(3, \mathbb{C})$ -action.

(2) $\mathcal{F}_{11} = \mathcal{R}_{11} = \mathcal{I}_{11} \sqcup \mathcal{S}_{11}$ and $\mathcal{F}_{12} = \mathcal{I}_{12} \sqcup \mathcal{S}_{12}$.

Proof.

(1) We may assume that \mathcal{A} has the description as in §4.6 and §4.7. By Theorem 2.4, we have $n_{\mathcal{A}_1, H} = 6$. Note that $F_{H,3}(\mathcal{A}_1) \leq 1$ since $M_2(\mathcal{A}) \subset H_{11}$ and $F_{H,4}(\mathcal{A}_1) \leq 1$ since $\mu_{\mathcal{A}_1, H} = 11 < 1 \cdot 4 + 4 \cdot 2$. Thus $F_H(\mathcal{A}_1) = [1, 5, 0, 0], [2, 3, 1, 0], [3, 2, 0, 1]$ or $[4, 0, 1, 1]$.

Suppose $F_H(\mathcal{A}_1) = [1, 5, 0, 0]$. By the description in §4.6, we have

$$M_1(\mathcal{A}) = \{\{1, 8\}, \{4, 8\}, \{4, 5\}, \{5, 10\}, \{1, 10\}\} \cup \{\{2, 6\}, \{2, 7\}, \{3, 7\}, \{3, 9\}, \{6, 9\}\}.$$

However, since one of the above two sets contains 3 elements of $H \cap M_1(\mathcal{A})$, H coincides with some $H_i \in \mathcal{A}$, which is a contradiction. Therefore $F_H(\mathcal{A}_1) \neq [1, 5, 0, 0]$.

Suppose $F_H(\mathcal{A}_1) = [2, 3, 1, 0]$. Note that the permutation $\rho = (1, 5, 8, 10, 4)(2, 6, 9, 3, 7)$ is an element of \mathfrak{S}_{11}^* , and the group $\langle \rho \rangle$ acts on $M_2(\mathcal{A})$ or $M_3(\mathcal{A})$ transitively. Since ρ is realized by the action of $\text{GL}(3, \mathbb{C})$ and $\text{Gal}(\mathbb{Q}[\sqrt{5}]/\mathbb{Q})$, we may assume $\{1, 9, 11\} \in H$, i.e., $H \parallel (x = 0)$. On the other hand, by the direct calculation, we have

$$M_1(\mathcal{A}) = \left\{ \begin{array}{lll} \{1, 8\} = (0, -\zeta), & \{1, 10\} = (0, \zeta + 1), & \{2, 6\} = (\zeta - 1, 2 - \zeta), \\ \{4, 5\} = (-\zeta, 0), & \{5, 10\} = (1 + \zeta, 0), & \{2, 7\} = (2 - \zeta, \zeta - 1), \\ \{3, 7\} = (\zeta - 1, 1), & \{3, 9\} = (1, 1), & \{4, 8\} = (\zeta + 1, \zeta + 1), \\ \{6, 9\} = (1, \zeta - 1), & & \end{array} \right\}.$$

It is easy to see that no three points of $M_1(\mathcal{A})$ share the same x -coordinate, which contradicts to $F_{H,2}(\mathcal{A}_1) = 3$. Therefore $F_H(\mathcal{A}_1) \neq [2, 3, 1, 0]$.

We show that, for each $\Gamma \in \{\{3, 2, 0, 1\}, \{4, 0, 1, 1\}\}$ and for each $P \in M_3(\mathcal{A})$, exists the unique line $H \notin \mathcal{A}$ such that $P \in H$ and $F_H(\mathcal{A}_1) = \Gamma$. First we assume $\{1, 2, 3, 4\} \in H$.

Suppose $\Gamma = \{3, 2, 0, 1\}$. Since $H \notin \mathcal{A}$, we have $H \cap M_1(\mathcal{A}) = \{\{5, 10\}, \{6, 9\}\}$, and hence $H = (x + (1 + \zeta)(y - 1) = 0)$. It is easy to check that $F_H(\mathcal{A}_1) = [3, 2, 0, 1]$.

Suppose $\Gamma = \{4, 0, 1, 1\}$. Since $H \notin \mathcal{A}$, we have $H \cap M_2(\mathcal{A}) = \{\{7, 8, 11\}\}$, and hence $H = (\zeta x - y + 1 = 0)$. It is easy to check that $F_H(\mathcal{A}_1) = [4, 0, 1, 1]$.

We have seen that the unique H exists for each Γ if $P = \{1, 2, 3, 4\}$. To have the unique H passing through another $P \in M_3(\mathcal{A})$, we have only to apply ρ repeatedly. Therefore, \mathcal{A}_1 's sharing the same $F_H(\mathcal{A}_1)$ are transferred by the $\text{GL}(3, \mathbb{C})$ -action.

Next we show $\mathcal{A}_1 \in \mathcal{I}_{12}$. Note that \mathcal{A}_1 in §4.8 satisfies $F_{H_{12}}(\mathcal{A}_1) = [3, 2, 0, 1]$ and $\mathcal{A}_1 \in \mathcal{I}_{12}$. It follows that $\mathcal{A}_1 \in \mathcal{I}_{12}$ when $F_H(\mathcal{A}_1) = [3, 2, 0, 1]$. We show that $\mathcal{A}_1 \in \mathcal{I}_{12}$ when $F_H(\mathcal{A}_1) = [4, 0, 1, 1]$. We may assume $\mathcal{A}_1 = \mathcal{A} \cup \{H\}$ with $H = (\zeta x - y + 1 = 0)$. By Theorem 2.4, we see that $\mathcal{A}_1 \in \mathcal{F}_{12}$ with $\exp(\mathcal{A}_1) = (1, 5, 6)$. We set $\mathcal{A}_2 = \mathcal{A}_1 \setminus \{H_5\}$ and $\mathcal{A}_3 = \mathcal{A}_2 \setminus \{H_9\}$. Since $n_{\mathcal{A}_1, H_5} = 6$, we have $\mathcal{A}_2 \in \mathcal{F}_{11}$ with $\exp(\mathcal{A}) = (1, 5, 5)$. Since $n_{\mathcal{A}_2, H_9} = 6$, we have $\mathcal{A}_3 \in \mathcal{F}_{10} = \mathcal{I}_{10}$. Therefore, we have $\mathcal{A}_2 \in \mathcal{I}_{11}$ and $\mathcal{A}_1 \in \mathcal{I}_{12}$. Thus we conclude that $\mathcal{A}_1 \in \mathcal{I}_{12}$ for both cases of $F_H(\mathcal{A}_1)$.

(2) Since $\mathcal{F}_{10} = \mathcal{I}_{10}$ and $\mathcal{S}_{11} \subset \mathcal{R}_{11}$, we have $\mathcal{F}_{11} = \mathcal{I}_{11} \sqcup \mathcal{S}_{11} = \mathcal{R}_{11}$ by Lemma 2.10. Let $\mathcal{A} \in \mathcal{F}_{12} \setminus \mathcal{S}_{12}$. Then, there exists $H \in \mathcal{A}$ such that $\mathcal{A}' = \mathcal{A} \setminus \{H\} \in \mathcal{F}_{11} = \mathcal{I}_{11} \sqcup \mathcal{S}_{11}$. If $\mathcal{A}' \in \mathcal{I}_{11}$, then $\mathcal{A} \in \mathcal{I}_{12}$. If $\mathcal{A}' \in \mathcal{S}_{11}$, we also have $\mathcal{A} \in \mathcal{I}_{12}$ by (1). Thus $\mathcal{F}_{12} = \mathcal{I}_{12} \sqcup \mathcal{S}_{12}$. \square

5. DETERMINATION OF \mathcal{S}_{12}

In this section, we show that \mathcal{S}_{12} consists of monomial arrangements associated to the group $G(4, 4, 3)$.

5.1. Realization of $\mathcal{A} \in \mathcal{S}_{12}$. Let $\mathcal{A} \in \mathcal{S}_{12}$. By Proposition 2.14, $F(\mathcal{A}) = [0, 16, 3]$.

We show that $F_H(\mathcal{A}) = [0, 4, 1]$ for any $H \in \mathcal{A}$. Note that $n_{\mathcal{A}, H} = 5$ for any $H \in \mathcal{A}$, since $\sum_{H \in \mathcal{A}} n_{\mathcal{A}, H} = 3 \cdot 16 + 4 \cdot 3 = 5 \cdot 12$. We have $F_{H,3}(\mathcal{A}) \leq 1$ for $H \in \mathcal{A}$, since $\mu_{\mathcal{A}, H} = 11 < 2 \cdot 3 + 3 \cdot 2$. If $M_3(H_0, \mathcal{A}) = \emptyset$ for some $H_0 \in \mathcal{A}$, then $11 = \mu_{P, H_0} = 2F_{H_0,2}(\mathcal{A})$, a contradiction. Thus, for any $H \in \mathcal{A}$, we have $F_{H,3}(\mathcal{A}) = 1$, and hence $F_H(\mathcal{A}) = [0, 4, 1]$.

We determine the realization of \mathcal{A} in $\mathbb{P}_{\mathbb{C}}^2$. We may set

$$M_3(\mathcal{A}) = \{\{1, 2, 3, 4\}, \{5, 6, 7, 8\}, \{9, 10, 11, 12\}\}.$$

Since $F_{H_9,2}(\mathcal{A}) = 4$, we may assume $M_2(H_9, \mathcal{A}) = \{\{1, 5, 9\}, \{2, 6, 9\}, \{3, 7, 9\}, \{4, 8, 9\}\}$.

We may set $\{1, 2, 3, 4\} \cup \{5, 6, 7, 8\} \notin \mathcal{A}$ as the infinity line H_{∞} and

$$h_1 = x, h_2 = x - 1, h_3 = x - p, h_4 = x - q, h_5 = y, h_6 = y - 1, h_7 = y - s, h_8 = y - t.$$

where $|\{0, 1, p, q\}| = |\{0, 1, s, t\}| = 4$. By choice of H_9 , we have $s = p, t = q$ and $h_9 = x - y$.

We set $\alpha_i = \prod_{j=0}^3 h_{4i-j}$ for $1 \leq i \leq 3$. Since $F_{H,2}(\mathcal{A}) = 4$ for any $H \in \mathcal{A}$, we have $V(\alpha_1, \alpha_2) = M_2(\mathcal{A}) \subset V(\alpha_3)$. Therefore we have $\alpha_3 \in \sqrt{(\alpha_1, \alpha_2)} = (\alpha_1, \alpha_2)$. Since $\deg(\alpha_i) = 4$ for $i = 1, 2, 3$, there exists $a, b \in \mathbb{C} \setminus \{0\}$ such that $\alpha_3 = a\alpha_1 + b\alpha_2$. Since $\alpha_3 \in (x - y)$, we have $b = -a$. Therefore we may set $a = 1, b = -1$. Now we obtain

$$\alpha_3 = \alpha_1 - \alpha_2 = x(x - 1)(x - p)(x - q) - y(y - 1)(y - p)(y - q).$$

Set $\beta = h_{10}h_{11}h_{12} = \alpha_3/(x - y)$. Then we have

$$\beta = x^3 + x^2y + xy^2 + y^3 - (p + q + 1)(x^2 + xy + y^2) + (pq + p + q)(x + y) - pq.$$

Since β is a symmetric polynomial in x and y , we may set

$$\beta = u^{-1}(x + uy + v)(ux + y + v)(x + y - 2w). \quad (u, v, w, \in \mathbb{C}, u \neq 0).$$

Then we have $\{9, 10, 11, 12\} = (w, w)$ and hence $v = -(1 + u)w$. Comparing coefficients of x^2y, x^2, x and constant terms, we obtain $1 = u + 1 + u^{-1}, -p - q - 1 = -w(u + 4 + u^{-1}), pq + p + q = 3u^{-1}(1 + u)^2w^2$ and $-pq = -2u^{-1}(1 + u)^2w^3$. Therefore we have

$$u^2 = -1, \quad p + q = 4w - 1, \quad pq + p + q = 6w^2, \quad pq = 4w^3.$$

and hence $2w - 1 = 0, \pm \sqrt{-1}$. Now it is easy to show that

$$\{1, p, q\} = \left\{1, \pm \sqrt{-1}, 1 \pm \sqrt{-1}\right\}, \left\{1, \frac{1 + \sqrt{-1}}{2}, \frac{1 - \sqrt{-1}}{2}\right\}.$$

These 3 possibilities of (p, q) are identified by the actions $(x, y) \mapsto (x/p, y/p)$ or $(x, y) \mapsto (x/q, y/q)$, which corresponds to the changing of scale so as to set H_2, H_3 or H_4 to be $(x = 1)$. Here we adopt $\{p, q\} = \{\sqrt{-1}, 1 + \sqrt{-1}\}$. Then we have

$$\alpha_3 = (x - y)(x - \sqrt{-1}y - 1)(x + y - 1 - \sqrt{-1})(x + \sqrt{-1}y - \sqrt{-1}).$$

Now we have obtained the unique realization of $\mathcal{A} \in \mathcal{S}_{12}$ up to the $\text{GL}(3, \mathbb{C})$ -action.

5.2. Verifying $\mathcal{A} \in \mathcal{S}_{12} \subset \mathcal{R}_{12}$. We set $\mathcal{A}_1 = \mathcal{A} \cup \{H_\infty\}$. It is easy to see that $n_{\mathcal{A}_1, H_\infty} = 6$. Since $\mu_{\mathcal{A}_1} = \mu_{\mathcal{A}} + 6$, we have $\chi(\mathcal{A}_1, t) = (t-1)(t-5)(t-7)$. Thus $\mathcal{A}_1 \in \mathcal{F}_{13}$ with $\exp(\mathcal{A}_1) = (1, 5, 7)$ by Theorem 2.8, and hence $\mathcal{A} \in \mathcal{S}_{12}$. We set $\mathcal{A}_2 = \mathcal{A}_1 \setminus \{H_9\}$ and $\mathcal{A}_3 = \mathcal{A}_2 \setminus \{H_{10}\}$. Since $n_{\mathcal{A}_1, H_9} = 6$, we have $\mathcal{A}_2 \in \mathcal{F}_{12}$ with $\exp(\mathcal{A}_2) = (1, 5, 6)$. Since $n_{\mathcal{A}_2, H_{10}} = 6$, we have $\mathcal{A}_3 \in \mathcal{F}_{11} = \mathcal{R}_{11}$. Therefore, $\mathcal{A}_2 \in \mathcal{R}_{12}$, $\mathcal{A}_1 \in \mathcal{R}_{13}$ and $\mathcal{A} \in \mathcal{R}_{12}$.

In fact, to check whether $\mathcal{A} \in \mathcal{F}_{12}$ belongs to \mathcal{S}_{12} or not, we have only to check $F(\mathcal{A})$.

Lemma 5.1. *If $\mathcal{A} \in \mathcal{F}_{12}$ satisfies $F(\mathcal{A}) = [0, 16, 3]$, then $\mathcal{A} \in \mathcal{S}_{12}$.*

Proof. If $\mathcal{A} \notin \mathcal{S}_{12}$, there exists $H_1 \in \mathcal{A}$ such that $n_{\mathcal{A}, H_1} \geq 6$. Since $\sum_{H \in \mathcal{A}} n_{\mathcal{A}, H} = 5 \cdot 12$, there exists $H_2 \in \mathcal{A}$ such that $n_{\mathcal{A}, H_2} \leq 4$. Since $F(\mathcal{A}) = [0, 16, 3]$, we have $F_{H_2}(\mathcal{A}) = [0, 1, 3]$. Take $H_3 \in \mathcal{A} \setminus \{H_2\}$ such that $\mu_{H_2 \cap H_3}(\mathcal{A}) = 2$. Then, since $H_3 \cap M_3(\mathcal{A}) = \emptyset$, we have $12 - 1 = \mu_{\mathcal{A}, H_3} = 2n_{\mathcal{A}, H_3}$, a contradiction. \square

Definition 5.2. An arrangement in \mathbb{C}^3 is called a monomial arrangement associated to the group $G(4, 4, 3)$ (see B.1 of [OT]), if it is $\text{GL}(3, \mathbb{C})$ -equivalent to $(\varphi_{4,4,3} = 0)$, where

$$\varphi_{4,4,3} = (x^4 - y^4)(y^4 - z^4)(z^4 - x^4).$$

It is easy to see that $\mathcal{A} = (\varphi_{4,4,3} = 0)$ satisfies $F(\mathcal{A}) = [0, 16, 3]$. Therefore,

$$\mathcal{S}_{12} = \{\text{monomial arrangements associated to the group } G(4, 4, 3)\} \subset \mathcal{R}_{12}.$$

By Proposition 4.3, we have $\mathcal{F}_{12} = \mathcal{S}_{12} \sqcup \mathcal{I}_{12} = \mathcal{R}_{12}$. Thus Theorem 1.1 holds for $|\mathcal{A}| = 12$.

6. EXAMPLE IN $\mathcal{F}_{13} \setminus \mathcal{R}_{13}$

In this section, we construct the example $\mathcal{A} \in \mathcal{F}_{13} \setminus \mathcal{R}_{13}$.

6.1. Defining equation of \mathcal{A} . Let $\mathcal{A}_0 = \{H_1, \dots, H_{12}\}$ be the line arrangement in \mathbb{C}^2 , where H_i is defined by h_i below for each $1 \leq i \leq 12$, with a generic parameter $\lambda \in \mathbb{C}$.

$$\begin{aligned} h_1 &= -\sqrt{3}x - y + \lambda + 1, \\ h_2 &= 2y + \lambda + 1, \\ h_3 &= \sqrt{3}x - y + \lambda + 1, \\ h_4 &= \sqrt{3}x - y + \lambda - 2, \\ h_5 &= -\sqrt{3}x - y + \lambda - 2, \\ h_6 &= 2y + \lambda - 2, \\ h_7 &= 2y - 2\lambda + 1, \\ h_8 &= \sqrt{3}x - y - 2\lambda + 1, \\ h_9 &= -\sqrt{3}x - y - 2\lambda + 1, \\ h_{10} &= (\lambda + 1)y + \sqrt{3}(1 - \lambda)x - \lambda^2 + \lambda - 1, \\ h_{11} &= \sqrt{3}\lambda x + (\lambda - 2)y - \lambda^2 + \lambda - 1, \\ h_{12} &= (1 - 2\lambda)y - \sqrt{3}x - \lambda^2 + \lambda - 1. \end{aligned}$$

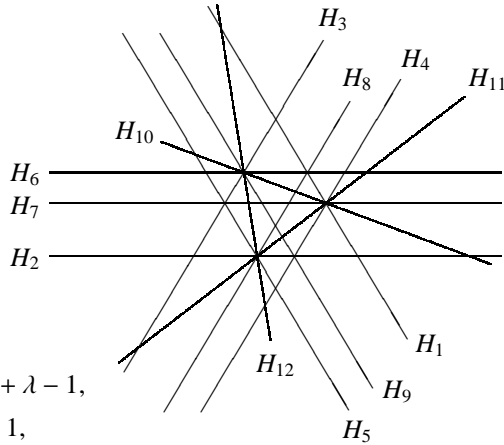


FIGURE 1. \mathcal{A}_0 with $\lambda = 2/3$

Note that $H_{3i-2} \mapsto H_{3i-1} \mapsto H_{3i}$ is obtained by the rotation with angle $-2\pi/3$. We will show that the cone $\mathcal{A} = c\mathcal{A}_0$ of \mathcal{A}_0 satisfies $\mathcal{A} \in \mathcal{F}_{13} \setminus \mathcal{R}_{13}$. Set $\mathcal{A} = \mathcal{A}_0 \cup \{H_{13}\}$ where H_{13} is the infinity line H_∞ . By calculation (or by reading off from the figure), we see that

$$\begin{aligned} M_2(\mathcal{A}) &= \{\{1, 6, 8\}, \{2, 4, 9\}, \{3, 5, 7\}\}, \\ M_3(\mathcal{A}) &= \{\{1, 5, 9, 13\}, \{2, 6, 7, 13\}, \{3, 4, 8, 13\}\} \subset H_{13}, \\ M_4(\mathcal{A}) &= \{\{1, 4, 7, 10, 11\}, \{2, 5, 8, 11, 12\}, \{3, 6, 9, 10, 12\}\}. \end{aligned}$$

Other intersection points of \mathcal{A} form $M_1(\mathcal{A})$. It follows that $F(\mathcal{A}) = [21, 3, 3, 3]$, $n_{\mathcal{A},H} = 6$ for any $H \in \mathcal{A}$, and $\chi(\mathcal{A}, t) = (t-1)(t-6)^2$. Thus $\mathcal{A} \in \mathcal{S}_{13}$ provided $\mathcal{A} \in \mathcal{F}_{13}$.

6.2. Freeness of \mathcal{A} . We show that $\mathcal{A} \in \mathcal{F}_{13}$ in terms of Yoshinaga's criterion in [Y].

Let (\mathcal{A}'', m) be the Ziegler restriction of \mathcal{A} onto H_∞ . It is defined by

$$y^3(-\sqrt{3}x-y)^3(\sqrt{3}x-y)^3\{(\lambda+1)y+\sqrt{3}(1-\lambda)x\}\{\sqrt{3}\lambda x+(\lambda-2)y\}\{(1-2\lambda)y-\sqrt{3}x\}=0.$$

By the change of coordinates

$$u = y + \sqrt{3}x, \quad v = y - \sqrt{3}x,$$

the defining equation of (\mathcal{A}'', m) becomes

$$u^3v^3(u+v)^3(u+\lambda v)(u+v-\lambda u)(\lambda u+\lambda v-v)=0.$$

Now recall the following.

Proposition 6.1 ([Y]). *Let \mathcal{B} be a central arrangement in \mathbb{C}^3 , $H \in \mathcal{B}$ and let (\mathcal{B}'', m) be the Ziegler restriction of \mathcal{B} onto H . Assume that $\chi(\mathcal{B}, t) = (t-1)(t-a)(t-b)$, and $\exp(\mathcal{B}'', m) = (d_1, d_2)$. Then \mathcal{B} is free if and only if $ab = d_1d_2$.*

Also, recall that, for a central multiarrangement (C, m) in \mathbb{C}^2 with $\exp(C, m) = (e_1, e_2)$ ($e_1 \leq e_2$), it holds that $e_1 + e_2 = |m| = \sum_{H \in C} m(H)$ and $e_1 = \min_{d \in \mathbb{Z}} \{d \mid D(C, m)_d \neq 0\}$. This follows from the fact that $D(C, m)$ is a rank two free module. For example, see [A].

Now since $\chi(\mathcal{A}, t) = (t-1)(t-6)^2$, it suffices to show that every homogeneous derivation of degree five is zero.

Assume that $\theta \in D(\mathcal{A}'', m)$ is homogeneous of degree five and show that $\theta = 0$. To check it, first, let us introduce a submultiarrangement (\mathcal{B}, m') of (\mathcal{A}'', m) defined by

$$u^3v^3(u+v)^3=0.$$

The freeness of (\mathcal{B}, m') is well-known. In fact, we can give its explicit basis as follows.

$$\partial_1 = (u+2v)u^3\partial_u - (2u+v)v^3\partial_v, \quad \partial_2 = (u+3v)vu^3\partial_u + (3u+v)uv^3\partial_v.$$

So $\exp(\mathcal{B}, m') = (4, 5)$. Since $D(\mathcal{B}, m') \supset D(\mathcal{A}'', m)$, there are scalars $a, b, c \in \mathbb{C}$ such that

$$\theta = (au + bv)\partial_1 + c\partial_2.$$

The scalars a, b and c are determined by the tangency conditions to the three lines $u + \lambda v = 0$, $u + v - \lambda u = 0$, $\lambda u + \lambda v - v = 0$. Then a direct computation shows that a, b and c satisfy

$$\lambda(\lambda-1) \begin{pmatrix} \lambda & -1 & -\lambda(\lambda+1) \\ 1 & \lambda-1 & -(\lambda-1)(\lambda-2) \\ \lambda-1 & -\lambda & -\lambda(\lambda-1)(2\lambda-1) \end{pmatrix} \begin{pmatrix} (\lambda^2-\lambda+1)a \\ (\lambda^2-\lambda+1)b \\ c \end{pmatrix} = 0.$$

The above linear equations imply that

$$-\lambda(\lambda-1)(\lambda-2)(2\lambda-1)(\lambda+1)(\lambda^2-\lambda+1) \begin{pmatrix} (\lambda^2-\lambda+1)a \\ (\lambda^2-\lambda+1)b \\ c \end{pmatrix} = 0.$$

Since λ is generic, it holds that $a = b = c = 0$. Hence $\theta = 0$, that is to say, $D(\mathcal{A}'', m)_5 = 0$. Now apply Proposition 6.1 to show that \mathcal{A} is free with $\exp(\mathcal{A}) = (1, 6, 6)$.

We can also construct the basis of $D(\mathcal{A})$ explicitly and give an alternative proof of the freeness of \mathcal{A} by Theorem 2.1. However, we omit to describe it here because of its length.

6.3. Non-recursive freeness of \mathcal{A} . We assume that $\mathcal{A} \in \mathcal{R}_{13}$ and deduce the contradiction. Recall that $\mathcal{A} \in \mathcal{S}_{13}$ with $\exp(\mathcal{A}) = (1, 6, 6)$. Thus, there exists a line $H \subset \mathbb{P}_{\mathbb{C}}^2$ such that $\mathcal{A}_1 = \mathcal{A} \cup \{H\} \in \mathcal{F}_{14}$. Note that $n_{\mathcal{A}_1, H} = 7$ by Theorem 2.4.

Step 1. $F_{H,2}(\mathcal{A}_1) \leq 3$.

Assume that $|H \cap M_1(\mathcal{A})| \geq 4$. By the description of $M_i(\mathcal{A})$ for $2 \leq i \leq 4$, we have

$$M_1(\mathcal{A}) = \left\{ \begin{array}{cccccccc} \{1, 2\}, & \{2, 10\}, & \{4, 5\}, & \{5, 10\}, & \{7, 8\}, & \{8, 10\}, & \{10, 13\}, \\ \{2, 3\}, & \{3, 11\}, & \{5, 6\}, & \{6, 11\}, & \{8, 9\}, & \{9, 11\}, & \{11, 13\}, \\ \{3, 1\}, & \{1, 12\}, & \{6, 4\}, & \{4, 12\}, & \{9, 7\}, & \{7, 12\}, & \{12, 13\} \end{array} \right\}.$$

Note that the points in each column are transferred each other by the rule on indices except for 13 as $3i+1 \mapsto 3i+2 \mapsto 3i+3 \mapsto 3i+1$, which corresponds to the rotation of lines with angle $-2\pi/3$. Note that some row contains 2 points on H . By symmetry, we may assume

$$|H \cap \{\{1, 2\}, \{2, 10\}, \{4, 5\}, \{5, 10\}, \{7, 8\}, \{8, 10\}, \{10, 13\}\}| \geq 2.$$

Since $H \notin \mathcal{A}$, H is one of the following.

$$\begin{array}{cccccc} \overline{\{1, 2\}\{4, 5\}}, & \overline{\{1, 2\}\{5, 10\}}, & \overline{\{1, 2\}\{7, 8\}}, & \overline{\{1, 2\}\{8, 10\}}, & \overline{\{1, 2\}\{10, 13\}}, & \overline{\{2, 10\}\{4, 5\}}, \\ \overline{\{2, 10\}\{7, 8\}}, & \overline{\{4, 5\}\{7, 8\}}, & \overline{\{4, 5\}\{8, 10\}}, & \overline{\{4, 5\}\{10, 13\}}, & \overline{\{5, 10\}\{7, 8\}}, & \overline{\{7, 8\}\{10, 13\}} \end{array}$$

However, it follows by the direct calculation that $n_{\mathcal{A}_1, H} = 10$ if $\{10, 13\} \in H$ and $n_{\mathcal{A}_1, H} = 11$ if $\{10, 13\} \notin H$, both contradicting to $n_{\mathcal{A}_1, H} = 7$. Thus $F_{H,2}(\mathcal{A}_1) = |H \cap M_1(\mathcal{A})| \leq 3$.

Step 2. $\sum_{i=3}^5 F_{H,i}(\mathcal{A}_1) \leq 1$.

If $\sum_{i=3}^5 F_{H,i}(\mathcal{A}_1) \geq 2$, then there exist 2 points $P, Q \in \bigcup_{i=2}^4 M_i(\mathcal{A})$ such that $H = \overline{PQ}$. If $(\mu_P(\mathcal{A}), \mu_Q(\mathcal{A})) = (2, 3), (2, 4), (3, 3), (3, 4)$ or $(4, 4)$, we have $\overline{PQ} \in \mathcal{A}$ by the description of $M_i(\mathcal{A})$ for $2 \leq i \leq 4$. Thus we may assume $P, Q \in M_2(\mathcal{A})$. By symmetry, we may assume $H = \overline{\{1, 6, 8\}\{2, 4, 9\}}$. However, by calculation, we have $n_{\mathcal{A}_1, H} = 9 \neq 7$, a contradiction.

By the above steps, we see that $F_H(\mathcal{A}_1) = [3, 3, 0, 1]$ or $[4, 2, 0, 0, 1]$.

Step 3. The case of $F_H(\mathcal{A}_1) = [3, 3, 0, 1]$.

Since $|H \cap M_3(\mathcal{A})| = F_{H,4}(\mathcal{A}_1) = 1$, we may assume $\{2, 6, 7, 13\} \in H$ by symmetry. Thus H is parallel to x -axis. Set $B = \bigcup_{i=2,6,7,13} H_i$, where H_2, H_6, H_7 are parallel to x -axis and H_{13} is the infinity line. Then we have $|M_1(\mathcal{A}) \setminus B| = 9$. We can check by calculation that the y -coordinates of the points in $M_1(\mathcal{A}) \setminus B$ differ each other. Thus we have $F_{H,2}(\mathcal{A}_1) = |H \cap M_1(\mathcal{A})| \leq 2$, a contradiction.

Step 4. The case of $F_H(\mathcal{A}_1) = [4, 2, 0, 0, 1]$.

Since $|H \cap M_4(\mathcal{A})| = F_{H,5}(\mathcal{A}_1) = 1$, we may assume $Q = \{1, 4, 7, 10, 11\} \in H$ by symmetry. Since $H \notin \mathcal{A}$, we have $H \cap M_1(\mathcal{A}) \subset \{\{2, 3\}, \{5, 6\}, \{8, 9\}, \{12, 13\}\}$. Since $|H \cap M_1(\mathcal{A})| = F_{H,2}(\mathcal{A}_1) = 2$, we can take $P \in H \cap \{\{2, 3\}, \{5, 6\}, \{8, 9\}\}$. Then $H = \overline{PQ}$. For $P = \{2, 3\}, \{5, 6\}$ or $\{8, 9\}$, by calculation, we have $n_{\mathcal{A}_1, H} = 8 \neq 7$, a contradiction.

Therefore we conclude that $\mathcal{A} \notin \mathcal{R}_{13}$. Now the proof of Theorem 1.1 is completed.

Below is the code of Maple we used to check the calculations in §6.3.

```
# Rotate_Function
RF:=f->simplify(subs(
  {s=(-x-sqrt(3)*y)/2,t=(sqrt(3)*x-y)/2},subs({x=s,y=t},f) ));

# Seeds for equations by rotation
S:=[-3^(1/2)*x-y+A+1,3^(1/2)*x-y+A-2,2*y-2*A+1,
  -3^(1/2)*x*A+y*A+3^(1/2)*x+y-A^2+A-1];

# All equations
Eq:=map(f->collect(f,[x,y]),
  [seq([S[i],RF(S[i]),RF(RF(S[i]))][1],i=1..4)]));

# Intersection Points, Abbreviated version
IP:=(f,g)->simplify(subs(solve({f,g},{x,y}},{x,y})):
P:=(a,b)->IP(Eq[a],Eq[b]):
```

```

# Line through 2 points
H := (A, B)-> simplify(
  (B[1]-A[1])*y-(B[2]-A[2])*x+(A[1]*B[2]-A[2]*B[1]) ):

# number of intersections of a line T and Eq
C:=T->nops({simplify(map(
  f->solve(subs(y=simplify(solve(T,y)),f),x) ,Eq ))[]}) :

# number of intersect. of a line through (a,b),(c,d) and Eq
CC:=(a,b,c,d)->C(H(P(a,b),P(c,d))):

# number of intersect. of a line through (a,b),(10,13) and Eq
CCC:=(a,b)->C(Eq[10]+
  solve(subs({x=P(a,b)[1],y=P(a,b)[2]},Eq[10]+U),U) ):

# Verification for Step 1
CC(1,2,4,5) ; CC(1, 2,5,10); CC(1, 2,7,8); CC(1,2,8,10);
CCC(1,2)    ; CC(2,10,4,5) ; CC(2,10,7,8); CC(4,5,7, 8);
CC(4,5,8,10); CCC(4, 5)    ; CC(5,10,7,8); CCC(7,8)    ;

# Verification for Step 2
CC(1,6,2,4);

# Verification for Step 3
S:={map(f->P(f[])[2],[
  [3, 1], [3,11], [1,12], [4, 5], [5,10],
  [4,12], [8, 9], [8,10], [9,11]
  )[]}:
nops(S);

# Verification for Step 4
CC(1,4,2,3); CC(1,4,5,6); CC(1,4,8,9);

```

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